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On a Generalized Class of Minimax Inequalities

Ram U. Verma

Mathematical Sciences Division, International Publications, 12046 Coed Drive, Orlando, Florida 32826

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Some results on a generalized class of minimax inequalities based on the r–I–G–H-KKM mapping theorems in a G–H-space setting are presented. The r–I–G–H-KKM mappings represent a new class of KKM mappings in G–H-spaces as well as in the interval spaces. © 1999 Academic Press

1. INTRODUCTION

Minimax inequalities have applications to the solvability of problems arising from variational inequalities, optimization theory, potential theory, complementarity problems, differential equations, and others. Recently, Chang and Ma [1], and Chang and Xian [2], based on the notion of the generalized KKM mappings, generalized some nonempty intersection theorems and their applications to minimax inequalities to the case of an H-space, while Joo' and Kassay [3], using the concept of the KKM selections obtained number of nonempty intersection theorems and minimax inequalities with applications to saddle points and variational inequalities in a pseudoconvex space setting. The author [4, 5] obtained some nonempty intersection theorems involving the G–H-KKM type selections and G–H-KKM type mappings in a G–H-space setting along with their applications to minimax inequalities and minimax theorems, which can be applied to variational inequalities and saddle point existence theorems. The results, thus obtained, can be applied to a number of results in pseudo-convex spaces and H-spaces. For more details on variational inequalities and other associated generalizations, we refer to [3, 6–14].

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Here our aim is to establish a new class of generalized minimax inequalities and several other applications in a G–H-space setting. This class of generalized minimax inequalities is of interest in the sense that unlike in the case of [1, 2], the notion of the generalized H-convexity (abbreviated to G–H-convexity) of a set is not involved in the present formulations.

Let X be a topological space, let $P(X)$ denote the power set of X, and let $\langle X \rangle$ be a family of all finite subsets of X. Let Δ^n denote a standard $(n-1)$ simplex $\{e_1, \ldots, e_n\}$ of R^n.

DEFINITION 1.1. A triple $(X, H, \{p\})$ is called a G–H-space [7] if X is a topological space and $H: \langle X \rangle \to P(X) \setminus \{\emptyset\}$ is a mapping such that:

(i) For each $F, G \in \langle X \rangle$, there exists an $F_0 \subset F$ such that $F_0 \subset G$ implies $H(F_0) \subset H(G)$.

(ii) For $F = \{x_1, \ldots, x_n\} \in \langle X \rangle$, there is a continuous mapping p: $\Delta^n \to H(F)$ such that for $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, we have

$$p(\{e_{i_1}, \ldots, e_{i_k}\}) \subset H(\{x_{i_1}, \ldots, x_{i_k}\}),$$

where $\{x_{i_1}, \ldots, x_{i_k}\} \subset F$.

A subset D of X is called G–H-convex in X if for each $A \in \langle X \rangle$, there exists a subset A_1 of A such that $A_1 \subset D$ implies $H(A_1) \subset D$.

A subset K of X is said to be compactly closed in X if $K \cap L$ is closed in K for all compact subsets L of X.

DEFINITION 1.2. Let $(X, H, \{p\})$ be a G–H-space, Y a topological space, T: $X \to P(Y)$ a multivalued mapping, and r: $X \to X$ a function. T is said to be an r–I–G–H-KKM mapping if for each $\{x_1, \ldots, x_n\} \in \langle X \rangle$, there exists some $\{r(x_1), \ldots, r(x_n)\} \in \langle X \rangle$ such that for any subset $\{r(x_{i_1}), \ldots, r(x_{i_k})\}$ of $\{r(x_1), \ldots, r(x_n)\}$, we have

$$T\Big(H\big(\{r(x_{i_1}), \ldots, r(x_{i_k})\}\big)\Big) \subset \bigcup_{j=1}^{k} T(x_{i_j}) \qquad \text{for } \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}.$$

For $X = Y$, Definition 1.2 reduces to:

DEFINITION 1.3. Let $(X, H, \{p\})$ be a G–H-space and r: $X \to X$ a function. A mapping T: $X \to P(X)$ is r–I–G–H-KKM if for each $\{x_1, \ldots, x_n\} \in \langle X \rangle$, there exists some set $\{r(x_1), \ldots, r(x_n)\} \in \langle X \rangle$ such that for any subset $\{r(x_{i_1}), \ldots, r(x_{i_k})\}$ of $\{r(x_1), \ldots, r(x_n)\}$, we have

$$T\Big(H\big(\{r(x_{i_1}), \ldots, r(x_{i_k})\}\big)\Big) \subset \bigcup_{j=1}^{k} T(x_{i_j}).$$

For $r = I$, the identity mapping, we have

DEFINITION 1.4. Let $(X, H, \{p\})$ be a G–H-space. A mapping $T: X \to P(X)$ is called an I–G–H-KKM mapping if for each $\{x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_n\}$ such that

$$T\Big(H\big(\{x_{i_1}, \ldots, x_{i_k}\}\big)\Big) \subset \bigcup_{j=1}^{k} T(x_{i_j}).$$

Next, we give an example [2] of an I–G–H-KKM mapping in an interval space. A topological space X is called an interval space if there exists a mapping $[\cdot, \cdot]$: $X \times X \to$ {connected subsets of X} such that $\{x_1, x_2\} \subset [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$. Among the notable special cases of the interval space, we have Hausdorff topological vector spaces, contractible spaces, and connected spaces. Joo' and Kassay [3], by considering $X = [0, 1]$ in R and f: $X \to X$ defined by $f(x) = x + \frac{1}{2}$ for $x \in [0, \frac{1}{2}]$, $f(x) = x - \frac{1}{2}$ for $x \in (\frac{1}{2}, 1)$, and $f(0) = 1$, showed that Brouwer's fixed point theorem was not true for interval spaces when X is equipped with the following topology: $A \subset X$ is closed iff A is finite or $A = X$. Clearly, X is a compact topological space under this topology, where for each $x_1, x_2 \in X$, the interval joining x_1 and x_2 is the whole space X. Thus, every interval is closed and connected and contains its end points:

EXAMPLE 1.1. Let X be an interval space and Y a topological space. A mapping T: $X \to P(Y)$ is I-KKM if

$$T([x_1, x_2]) \subset \bigcup_{i=1}^{2} T(x_i) \qquad \text{for all } x_1, x_2 \in X.$$

2. GENERALIZED MINIMAX INEQUALITIES

In this section, we first recall an auxiliary result and then intend to prove the main minimax inequality theorem with its several special classes of minimax inequalities in a G–H-space setting.

LEMMA 2.1 [5]. *Let $(X, H, \{p\})$ be a G–H-space and let T: $X \to P(X) \setminus \{\varnothing\}$ be an r–I–G–H-KKM mapping. Let $T(H(A))$ be compact for all $A \in \langle X \rangle$, and let q: $H(F) \to T(H(F))$ be a continuous mapping for all $F = \{x_1, \ldots, x_n\} \in \langle X \rangle$. Suppose that*:

(i) *$T(x)$ is compactly closed in X for all $x \in X$.*

(ii) *There exists an $A \in \langle X \rangle$ such that $\bigcap_{x \in A} T(x)$ is a compact subset of X. Then $\bigcap_{x \in X} T(x) \neq \varnothing$.*

THEOREM 2.1. *Let $(X, H, \{p\})$ be a G–H-space, let $T: X \to P(X) \setminus \{\emptyset\}$ be any mapping such that $T(H(A))$ is compact for all $A \in \langle X \rangle$, and let $q: H(A) \to T(H(A))$ be a continuous function. Suppose that $f, g: X \times X \to R$ and $r: X \to X$ are functions such that*:

(i) $f(x, y) \leqq g(x, y)$ *for all* $(x, y) \in X \times X$.

(ii) $y \to f(x, y)$ *is lower semicontinuous on compact subsets of* X.

(iii) *For each* $A \in \langle X \rangle$, $\bigcap_{x \in A} \{y \in X: f(x, y) \leqq a\}$ *is a compact subset of* X *for all* $a \in R$.

(iv) *For each* $\{x_1, \dots, x_n\} \in \langle X \rangle$, *there exists a set* $\{r(x_1), \dots, r(x_n)\} \in \langle X \rangle$ *such that for any subset* $\{r(x_{i_1}), \dots, r(x_{i_k})\}$ *of* $\{r(x_1), \dots, r(x_n)\}$ *and for any* $x \in H(\{r(x_{i_1}), \dots, r(x_{i_k})\})$, *we have*

$$f(x, y) \geqq \min_{1 \leqq j \leqq k} \left\{ f(x_{i_j}, y), g(x_{i_j}, y) \right\} \qquad \textit{for all } y \in X.$$

Then one of the following statements holds:

(1) *There exists an element* $y_0 \in X$ *such that* $f(x, y_0) \leqq a$ *for all* $x \in X$.

(2) *There is an element* $x_0 \in X$ *such that* $g(x_0, x_0) > a$.

Proof. Let us define mappings $S, T: X \to P(X) \setminus \{\emptyset\}$, respectively, by

$$S(x) = \{ y \in X : g(x, y) \leqq a \text{ for all } x \in X \}$$

and

$$T(x) = \{ y \in X : f(x, y) \leqq a \text{ for all } x \in X \}.$$

Assume (2) does not hold. Then $g(x, x) \leqq a$ for all $x \in X$. As a result, $S(x)$ is nonempty. By (i), $S(x) \subset T(x)$. Next, to show that T is r–I–G–H-KKM, if for each $\{x_1, \dots, x_n\} \in \langle X \rangle$, there exists some set $\{(r(x_1), \dots, r(x_n)\} \in \langle X \rangle$ such that for any $x \in H(\{r(x_{i_1}), \dots, r(x_{i_k})\})$ and for some $y \in T(x)$, we have that $f(x, y) \leqq a$. Applying (iv), there exists some index m $(1 \leqq j \leqq k)$ such that $g(x_{i_m}, y) \leqq a$ or $f(x_{i_m}, y) \leqq a$. Thus, we have

$$y \in S(x_{i_m}) \subset \bigcup_{j=1}^{k} S(x_{i_j}) \qquad \text{or} \qquad y \in T(x_{i_j}) \subset \bigcup_{j=1}^{k} T(x_{i_j}),$$

so

$$y \in \bigcup_{j=1}^{k} S(x_{i_j}) \cup \bigcup_{j=1}^{k} T(x_{i_j}) \subset \bigcup_{j=1}^{k} T(x_{i_j}).$$

Therefore,

$$T\Big(H\big(\{r(x_{i_1}),\dots,r(x_{i_k})\}\big)\Big) \subset \bigcup_{j=1}^{k} T(x_{i_j}),$$

that is, T is an r–I–G–H-KKM mapping. Next, by (ii), each $T(x)$ is compactly closed in X. As now all the conditions of Lemma 2.1 are met, we have $\bigcap_{x\in X} T(x) \neq \emptyset$. Hence, there exists an element $y_0 \in X$ such that $f(x, y_0) \leqq a$ for all $x \in X$. This completes the proof of (1).

When X is compact in Theorem 2.1, we arrive at:

THEOREM 2.2. *Let $(X, H, \{p\})$ be a compact G–H-space, T: $X \to P(X)\setminus\{\emptyset\}$ any mapping such that $T(H(A))$ is compact for all $A \in \langle X\rangle$, and q: $H(A) \to T(H(A))$ a continuous mapping. Suppose that f, g: $X \times X \to R$ and r: $X \to X$ are functions such that:*

(i) *$f(x, y) \leqq g(x, y)$ for all $(x, y) \in X \times X$.*

(ii) *$y \to f(x, y)$ is lower semicontinuous.*

(iii) *For each $F = \{x_1, \dots, x_n\} \in \langle X\rangle$, there is a set $\{r(x_1), \dots, r(x_n)\} \in \langle X\rangle$ such that for any subset $\{r(x_{i_1}), \dots, r(x_{i_k})\}$ of $\{r(x_1), \dots, r(x_n)\}$ and for any $x \in H(\{r(x_{i_1}), \dots, r(x_{i_k})\})$, we have*

$$f(x,y) \geqq \min_{1\leqq j\leqq k} \big\{f(x_{i_j}, y), g(x_{i_j}, y)\big\} \qquad \textit{for all } y \in X.$$

Then there is an element $y_0 \in X$ such that $f(x, y_0) \leqq a$ for all $x \in X$ and for $a \in R$.

For $f = g$ in Theorem 2.2, we have

THEOREM 2.3. *Let $(X, H, \{p\})$ be a compact G–H-space, let T: $X \to P(X)\setminus\{\emptyset\}$ be any mapping such that $T(H(A))$ is compact for all $A \in \langle X\rangle$, and let q: $H(A) \to T(H(A))$ be a continuous function. Let f: $X \times X \to R$ and r: $X \to X$ be functions such that:*

(i) *$y \to f(x, y)$ is lower semicontinuous.*

(ii) *For each $\{x_1, \dots, x_n\} \in \langle X\rangle$, there exists a set $\{r(x_1), \dots, r(x_n)\} \in \langle X\rangle$ such that for each subset $\{r(x_{i_1}), \dots, r(x_{i_k})\}$ of $\{r(x_1), \dots, r(x_n)\}$ and for any $x \in H(\{r(x_{i_1}), \dots, r(x_{i_k})\})$, we have*

$$f(x,y) \geqq \min_{1\leqq j\leqq k} f(x_{i_j}, y) \qquad \textit{for all } y \in X.$$

Then there is an element $y_0 \in X$ such that $f(x, y_0) \leqq a$ for all $x \in X$ and for $a \in R$.

When $r = I$, the identity mapping, in Theorem 2.3, we arrive at:

THEOREM 2.4. *Let $(X, H, \{p\})$ be a compact G–H-space, $T: X \to P(X) \setminus \{\emptyset\}$ any mapping with $T(H(A))$ compact, and $q: H(A)) \to T(H(A))$ a continuous function for all $A \in \langle X \rangle$. Suppose that a function $f: X \times X \to R$ satisfies the following assumptions:*

(i) *$y \to f(x, y)$ is lower semicontinuous.*

(ii) *For each $\{x_1, \dots, x_n\} \in \langle X \rangle$, there exists a subset $\{x_{i1}, \dots, x_{ik}\}$ of $\{x_1, \dots, x_n\}$ such that for any $x \in H(\{x_{i_1}, \dots, x_{i_k}\})$, we have*

$$f(x, y) \geqq \min_{1 \leqq j \leqq k} f(x_{i_j}, y) \qquad \text{for all } y \in X.$$

Then there exists an element $y_0 \in X$ such that $f(x, y_0) \leqq a$ for all $x \in X$ and for $a \in R$.

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